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Spectral partitions on infinite graphs

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Abstract. Statistical models on infinite graphs may exhibit inhomogeneous thermodynamic behaviour at macroscopic scales. This phenomenon is of a geometrical origin and may be properly described in terms of *spectral partitions* into subgraphs with well defined spectral dimensions and spectral weights. These subgraphs are shown to be thermodynamically homogeneous and effectively decoupled.

1. Introduction

The study of model systems without translation invariance is an interesting and complex subject of modern statistical mechanics. A very general description of this situation is in terms of statistical models on graphs, that is on generic networks formed by sites, where dynamical variables reside, and links connecting pairwise sites whose variables are coupled. This is the direct extension of the typical set-up valid for crystalline lattices, which are indeed very special, homogeneous graphs.

On the other hand, graphs are not in general homogeneous and the main question is how these inhomogeneities affect physical properties and give rise to relevant changes with respect to lattices. While small-scale inhomogeneities will affect local properties, one expects that only large-scale inhomogeneities are relevant for bulk thermodynamic properties. Most likely, the latter properties are those that show universal features which depend only on a few global parameters, just as in the case of lattices. The study of such universality requires consideration of infinite graphs (with certain natural restrictions given below), where the thermodynamic limit is taken.

The main relevant geometrical parameter affecting universal properties is the spectral dimension $\bar{d}$ of an infinite graph $\mathcal{G}$ [1–3]. It generalizes the Euclidean dimension of lattices to arbitrary real values and is naturally defined from the infrared behaviour of the spectral density of the Laplacian operator on $\mathcal{G}$ [3]. An equivalent definition, which is adopted in this work, is in terms of average properties of random walks on $\mathcal{G}$ at large times, that is to say of the singularities of the Gaussian model on the same graph [2, 3].

On the other hand, the spectral dimension of the whole graph $\mathcal{G}$, by itself turns out not to be sensitive to macroscopic inhomogeneities strong enough to give rise to true thermodynamic inhomogeneities. Indeed, it may happen that distinct macroscopic parts of an infinite graph exhibit distinct thermodynamic behaviour. We shall show below that

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such parts can be characterized in terms of their own spectral dimension, possibly plus a spectral weight, resulting in an effective *spectral partition* of $\mathcal{G}$. The crucial point is that these parts form subgraphs which are thermodynamically *independent*, that is to say completely uncoupled as far as thermodynamic properties are concerned. In other words, inhomogeneous thermodynamic behaviour on the same infinite graph necessarily implies effective decoupling.

2. Infinite graphs: basic definitions, measure and averages

A (unoriented) graph $\mathcal{G}$ (see, for instance, the classic book [4]) is the ordered couple (G, G_L) formed by a countable set G of vertices (or sites, or nodes), that we shall generically indicate with lowercase Latin letters, $i, j, k, \ldots,$ and a set G_L of unoriented links (or bonds) which connect pairwise the sites and are therefore naturally denoted by pairs $(i, j) = (j, i)$. When the set G is finite, $\mathcal{G}$ is a *finite* graph and we shall denote by N the number of vertices of $\mathcal{G}$. A subgraph $\mathcal{G}'$ of $\mathcal{G}$ is a graph such that $G' \subseteq G$ and $G'_L \subseteq G_L$. A subgraph is said to be *complete* if it has all the available links, that is if, given the subset of nodes G', the subset of links G'_L is the largest possible one.

A path in $\mathcal{G}$ is a sequence of consecutive links $\{(i, k)(k, h) \ldots (n, m)(m, j)\}$. A graph is said to be connected, if for any two points $i, j \in G$ there is always a path joining them. In the following we will consider only connected graphs.

The graph topology can be described algebraically by its adjacency matrix $\boldsymbol{A}$ with elements

$$A_{ij} = \begin{cases} 1 & \text{if} \quad (i, j) \in G_L \\ 0 & \text{if} \quad (i, j) \notin G_L. \end{cases} \tag{2.1}$$

The Laplacian matrix $\boldsymbol{L}$ on the graph $\mathcal{G}$ has elements

$$L_{ij} = z_i \delta_{ij} - A_{ij} \tag{2.2}$$

where $z_i = \sum_j A_{ij}$, the number of nearest neighbours of i, is called the coordination number (or degree) of site i. Here we will consider graphs with $z_{\max} = \sup_i z_i < \infty$.

One can also consider a generalization of the adjacency matrix, which corresponds to the ferromagnetic and uniformly bounded coupling J_{ij}, with $J_{ij} \neq 0 \Longleftrightarrow A_{ij} = 1$ and $\sup J_{ij} < \infty$, $\inf J_{ij} > 0$. The elements of the generalized Laplacian matrix then read

$$\mathcal{L}_{ij} = J_i \delta_{ij} - J_{ij} \tag{2.3}$$

where $J_i = \sum_j J_{ij}$.

Every connected graph $\mathcal{G}$ is endowed with an intrinsic metric generated by the chemical distance $r_{i,j}$ which is defined as the number of links in the shortest path(s) connecting vertices i and j.

Let us now consider thermodynamic averages on infinite graphs (for an up to date mathematical overview on infinite graphs, see [5]). The Van Hove sphere $S_{o,r} \subset \mathcal{G}$ of centre $o \in G$ and radius r is the complete subgraph of $\mathcal{G}$ containing all $i \in G$ whose distance from o is $\leqslant r$ and all the links of $\mathcal{G}$ joining them. We will call $N_{o,r}$ the number of vertices contained in $S_{o,r}$.

In the thermodynamic limit the average $[f]_G$ of a real-valued function f on G is

$$[f]_G \equiv \lim_{r\to\infty} \frac{1}{N_{o,r}} \sum_{i \in S_{o,r}} f_i. \tag{2.4}$$

This average does not depend on the choice of the origin $o \in G$ provided f is bounded from below and

$$\lim_{r\to\infty} \frac{|\partial S_{o,r}|}{N_{o,r}} = 0 \tag{2.5}$$

where $|\partial S_{o,r}|$ is the number of vertices of the sphere $S_{o,r}$ connected with the rest of the graph [6]. Here we shall restrict our attention to graphs with this property.

The measure $|A|$ of a subset $A \subset G$ is the average value $[\chi(A)]_G$ of its characteristic function $\chi_i(A)$ defined by $\chi_i(A) = 1$ if $i \in A$ and $\chi_i(A) = 0$ if $i \notin A$. The measure of a subset of links $G'_L \subseteq G_L$ is similarly given by

$$|G'_L| \equiv \lim_{r\to\infty} \frac{N'_{L,r}}{N_{o,r}} \tag{2.6}$$

where $N'_{L,r}$ is the number of links of G'_L contained in the sphere $S_{o,r}$. Any two non-zero-measure subsets A and B of G are said to be equivalent if their symmetric difference has zero measure, that is $|A| = |B| = |A \cap B|$. For any given non-zero-measure subsets $A \subset G$ we shall denote its equivalence class by $\{A\}$. Then A is said to be a representative of $\{A\}$. With the subgraph $\mathcal{G}'$ defined by the ordered double (G', G'_L), we identify the measure of the subgraph as the measure $|G'|$ of its points.

Given a (non-zero-measure) subset $A \subset G$, we define the average on A of any real-valued function f on G as

$$[f]_A = [\chi(A)\, f]_G. \tag{2.7}$$

By definition $[f]_A$ is a function only of the equivalence classes, that is $[f]_A = [f]_{\{A\}}$. Moreover, quite evidently $[f]_C = [f]_A + [f]_B$ whenever $C = A \cup B$ and $|A \cap B| = 0$.

Given a complete subgraph $\mathcal{M} = (M, M_L)$, we denote by $\bar{\mathcal{M}}$ its complement in $\mathcal{G}$. This is formed by all points that do not belong to M and by all links of G_L which connect them. $\bar{\mathcal{M}}$ is therefore a complete subgraph. We call the pair $(\mathcal{M}, \bar{\mathcal{M}})$ a partition of order two of $\mathcal{G}$ whenever both M and its complement $\bar{M}$ are non-zero-measure subsets of G.

We now introduce the important concept of the *minimal distance* $\underline{D}(\mathcal{A}, \mathcal{B})$ between any pair $\mathcal{A}$, $\mathcal{B}$ of non-zero-measure subgraphs of $\mathcal{G}$ such that $|A \cap B| = 0$. It is defined as

$$\underline{D}(\mathcal{A}, \mathcal{B}) = \min(n : \ |A \cap_n B| > 0) \tag{2.8}$$

where

$$A \cap_n B = \{i \in A : \ \mathrm{dist}(i, B) = n\} \qquad \mathrm{dist}(i, B) = \min_{j\in B} r_{i,j}. \tag{2.9}$$

For $n = 0$, $\cap_n$ reduces to the usual intersection operator. Note that, while in general the relation $A \cap_n B$ is not symmetric in A, B, the minimal distance is symmetric: $\underline{D}(\mathcal{A}, \mathcal{B}) = \underline{D}(\mathcal{B}, \mathcal{A})$. In fact, from the boundedness of z_i, it can be shown by induction on n that

$$|B \cap_n A| \geqslant (z_{\max})^{-n}\, |A \cap_n B| \tag{2.10}$$

so that

$$|A \cap_n B| > 0 \quad \Longrightarrow \quad |B \cap_n A| > 0 \tag{2.11}$$

implying our assertion.

Consider now the minimal distance between the two members of a partition of order two. Suppose $\underline{D}(\mathcal{M}, \bar{\mathcal{M}}) = n > 1$; then $|M \cap_n \bar{M}| > 0 \Rightarrow |M \cap_{n-1} \bar{M}| > 0$ from the boundedness of z_i. This implies that if $\underline{D}(\mathcal{M}, \bar{\mathcal{M}})$ is finite, then $\underline{D}(\mathcal{M}, \bar{\mathcal{M}}) = 1$. In this case we may say that $\mathcal{M}$ and $\bar{\mathcal{M}}$ are *densely interlaced*, while in the opposite case that they are *infinitely separated*. From the definition of minimal distance, it follows that if two subgraphs $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{G}$ are infinitely separated, their common boundary $\partial(\mathcal{A}, \mathcal{B})$ (i.e. the links $(i, j) \in G_L$ with $i \in A$ and $j \in B$) is a zero-measure set. Then the two subgraphs can be disconnected by cutting such a zero-measure set of links. This relates the property of infinite separability to the simple separability property defined in [6]. Indeed, the two definitions coincide. We shall term a *separable partition* a partition $(\mathcal{M}, \bar{\mathcal{M}})$ where $\mathcal{M}$ and $\bar{\mathcal{M}}$ are infinitely separated.

3. The Gaussian model: infrared behaviour and the spectral dimension

The Gaussian model on $\mathcal{G}$ is defined [2] by assigning a real-valued random variable ϕ_i to each node $i \in G$ and then prescribing the following probability measure:

$$\mathrm{d}\mu_r[\phi] = \frac{1}{Z_r} \exp\left[-\sum_{i,j \in S_{o,r}} \phi_i (\boldsymbol{L} + m^2 \boldsymbol{\eta})_{ij} \phi_j\right] \prod_{i \in S_{o,r}} \mathrm{d}\phi_i \qquad (3.1)$$

for the collection $\phi = \{\phi_i;\ i \in S_{o,r}\}$. Here Z_r is the proper normalization factor, $m > 0$ is a free parameter and $\boldsymbol{\eta}$ is the diagonal matrix with elements $\eta_{ij} = \eta_i \delta_{ij}$ with the real numbers η_i positive definite and uniformly bounded throughout G (that is, $0 < \eta_{\min} \leqslant \eta_i \leqslant \eta_{\max}$, $\forall i \in G$).

The thermodynamic limit is achieved by letting $r \to \infty$ and defines a Gaussian measure over the entire $\phi = \{\phi_i;\ i \in G\}$ which does not depend on the centre of the Van Hove sphere o [6]. The covariance of this Gaussian process reads

$$\langle \phi_i \phi_j \rangle \equiv C_{ij}(m^2) = (\boldsymbol{L} + m^2 \boldsymbol{\eta})^{-1}_{ij} \qquad (3.2)$$

and hence it satisfies by definition the Schwinger–Dyson (SD) equation

$$(J_i + m^2 \eta_i) C_{ij}(m^2) - \sum_{k \in G} J_{ik} C_{kj}(m^2) = \delta_{ij}. \qquad (3.3)$$

Setting

$$C_{ij} = \frac{(1 - W)^{-1}_{ij}}{J_i + m^2 \eta_i} \qquad W_{ij} = \frac{J_{ij}}{J_j + m^2 \eta_j} \qquad (3.4)$$

one obtains the standard connection with the random walk (RW) over $\mathcal{G}$ [2]:

$$(1 - W)^{-1}_{ij} = \sum_{t=0}^{\infty} (W^t)_{ij} = \sum_{\gamma : i \leftarrow j} W[\gamma] \qquad (3.5)$$

where the last sum runs over all paths from j to i, each weighted by the product along the path of the one-step probabilities in W:

$$\gamma = (i, k_{t-1}, \ldots, k_2, k_1, j) \quad \Longrightarrow \quad W[\gamma] = W_{ik_{t-1}} W_{k_{t-1}k_{t-2}}, \ldots, W_{k_2 k_1} W_{k_1 j}. \qquad (3.6)$$

Note that, as long as $m > 0$, we have $\sum_i (W^t)_{ij} < 1$ for any t, namely the walker has a non-zero death probability. This implies that C_{ij} is a smooth function of m^2 for $m \geqslant \epsilon > 0$. In the limit $m \to 0$ the walker never dies and the sum over paths in equation (3.5) is dominated by the infinitely long paths which sample the large-scale structure of the entire graph (here 'large scale' refers to the metric induced by the chemical distance alone). This typically reflects itself into a singularity of C_{ij} at $m = 0$ whose nature does not depend on the detailed form of J_{ij} or η_i, as long these stay uniformly positive and bounded.

Of particular importance is the leading singular infrared behaviour, as $m^2 \to 0$, of the average $[C(m^2)]_G$ of $C_{ii}(m^2)$, which is a positive-definite quantity, over all points i of the graph $\mathcal{G}$, which we may write in general as

$$\mathrm{Sing}[C(m^2)]_G \sim c(m^2)^{\bar{d}/2-1}. \qquad (3.7)$$

The parameter $\bar{d}$ is called the spectral dimension of the graph $\mathcal{G}$ and on regular lattices it coincides with the usual Euclidean dimension. Henceforth we shall call the coefficient c in equation (3.7) the *spectral weight*. The name *spectral dimension* is related to the behaviour of the spectral density $\rho(l)$ of low-lying eigenvalues of the Laplacian $\boldsymbol{L}$; indeed, it can be shown [3] that $\rho(l)$ scales as a power of l for $l \to 0$, that is $\rho(l) \sim l^{\bar{d}/2-1}$.

4. Large-scale inhomogeneity: homogeneity classes and spectral classes

In the study of statistical models one often has to deal with the average $[C(m^2)]_A$ of $C_{ii}(m^2)$ over a generic positive measure subset $A \subset G$ and, in particular, one has to consider the leading singular behaviour of $[C(m^2)]_A$ as $m^2 \to 0$. On regular lattices this singular behaviour is independent of A and it actually coincides with that obtained averaging over all points of $\mathcal{G}$:

$$\mathrm{Sing}[C(m^2)]_A = \mathrm{Sing}[C(m^2)]_G \qquad \forall\, A \subset G \quad |A| > 0. \qquad (4.1)$$

This property arises from the large-scale homogeneity of regular lattices due to translation invariance. On graphs, where translation invariance is lost, this property can still hold if the inhomogeneity is limited to finite scales. More generally, it may happen that inhomogeneity extends to large scales and the singular parts of equation (4.1) are different on different subsets. However, we will prove that such subsets must satisfy very strong topological constraints: a large-scale inhomogeneous graph always consists of homogeneous parts joined together by a zero-measure set of links. Therefore, the splitting of infrared behaviour always corresponds to a macroscopically evident inhomogeneity of the graph.

In this section we will give a rigorous formulation of these statements through the following steps.

- Let us suppose that the graph $\mathcal{G}$ has indeed a large-scale inhomogeneity that manifests itself through the existence of at least one non-zero-measure subset $A \subset G$ such that, as $m^2 \to 0$,

$$\mathrm{Sing}[C(m^2)]_A \sim c_A(m^2)^{\bar{d}_A/2-1} \qquad (4.2)$$

  with $\bar{d}_A \neq \bar{d}$.
- We then define $M \subset G$ to be a *maximally homogeneous* (or more briefly *maximal*) subset with respect to $\bar{d}_A$ whenever:
  (a) $|M \cap A| > 0$;
  (b) $\mathrm{Sing}[C(m^2)]_M \sim c_M(m^2)^{\bar{d}_M/2-1}$, with $\bar{d}_M = \bar{d}_A$;
  (c) for any non-zero-measure subset $B \subset M$ we have $\bar{d}_B = \bar{d}_M$;
  (d) there exists no $B \supset M$ such that $\bar{d}_B = \bar{d}_M$ and $|B| > |M|$.

  By this definition it follows that the set of all maximal subsets with respect to $\bar{d}_M$ coincides with the equivalence class $\{M\}$ and we will call it the *homogeneity class* of $\bar{d}_M$.
- Next we prove

  Theorem 1. *The subgraphs $\mathcal{M}$ and its complement $\bar{\mathcal{M}}$ are infinitely separated, i.e. their minimal distance $\underline{D}(\mathcal{M}, \bar{\mathcal{M}})$ is infinite and they define a separable partition of $\mathcal{G}$. Since this separability is induced by the spectral properties embodied by the spectral dimension, we call this a spectral partition (of order two) of $\mathcal{G}$.*

- Finally, we consider a Gaussian model on the graph $\mathcal{M}$ showing that, from the infinite separability of $\mathcal{M}$ and $\bar{\mathcal{M}}$ the spectral dimension of $\mathcal{M}$ is $\bar{d}_M$. Therefore, $\bar{d}_M$ is a property of the graph $\mathcal{M}$ and defines a *spectral class*. This chain of arguments may now be applied to $\bar{\mathcal{M}}$, splitting off a new spectral class if $\bar{\mathcal{M}}$ has a large-scale inhomogeneity of the type given above. The process can be repeated until necessary, yielding a complete spectral partition of the original graph $\mathcal{G}$ into spectral classes.

Proof of theorem 1. Let us suppose *ad absurdum* that $\underline{D}(\mathcal{M}, \bar{\mathcal{M}}) = 1$ and therefore that there exists a non-zero-measure subset $\bar{M}' \subset \bar{M}$ such that $\underline{D}(\mathcal{M}, \bar{\mathcal{M}}') = 1$. From the maximality of M it follows that $\bar{d}_M \neq \bar{d}_{\bar{M}'}$. Let us consider the random-walk representation (3.5) of $C_{ii}(m^2)$ with $i \in \bar{M}'$:

$$C_{ii}(m^2) = \frac{1}{J_i + m^2\eta_i} \sum_{\gamma:i\leftarrow i} W[\gamma]. \tag{4.3}$$

Next consider a site $k \in M$ whose distance from i is 1. This site exists from the hypothesis $\underline{D}(\mathcal{M}, \bar{\mathcal{M}}') = 1$. Then, from the sum over paths in the left-hand side of (4.3) let us retain only the paths containing k. Then, from the boundedness and positivity of J_{ij} and η_i one obtains

$$C_{ii}(m^2) \geqslant \frac{C_{kk}(m^2)}{J_{\max} + m^2\eta_{\max}}. \tag{4.4}$$

Averaging over M and then over $\bar{M}'$ we obtain

$$[C(m^2)]_{\bar{M}'} \geqslant K\,[C(m^2)]_M \tag{4.5}$$

where K is a positive constant. Now, taking $m^2 \to 0$ and using the asymptotic expression for $[C(m^2)]$ given in (3.7) we obtain

$$(m^2)^{\bar{d}_{\bar{M}'}/2-1} \geqslant K'\,(m^2)^{\bar{d}_M/2-1}. \tag{4.6}$$

Since this argument applies equally well with $\mathcal{M}$ and $\bar{\mathcal{M}}$ interchanged, one obtains

$$(m^2)^{\bar{d}_M/2-1} \geqslant K''\,(m^2)^{\bar{d}_{\bar{M}'}/2-1} \tag{4.7}$$

which gives $\bar{d}_M = \bar{d}_{\bar{M}'}$, contradicting the hypothesis. Therefore, $\underline{D}(\mathcal{M}, \bar{\mathcal{M}}) = \infty$ and $\mathcal{M}$ and $\bar{\mathcal{M}}$ must be infinitely separated. □

The infinite separability of $\mathcal{M}$ and $\bar{\mathcal{M}}$ implies that the two subgraphs can be disconnected by cutting a zero-measure set of links. This very peculiar property implies *thermodynamic independence*, that is the decoupling, in the thermodynamic limit, of a model defined on the whole graph $\mathcal{G}$ into two models defined independently on $\mathcal{M}$ and $\bar{\mathcal{M}}$ [6].

This applies in particular to the Gaussian model, so that the two averages of $C_{ii}(m^2)$ on $\mathcal{M}$ and $\bar{\mathcal{M}}$ are independent quantities, each satisfying a relation like equation (3.7) with two distinct spectral dimensions. Most importantly, to any non-zero-measure subset of $\mathcal{M}$ there corresponds by construction the same spectral dimension $\bar{d}$ of $\mathcal{M}$. We can say then that $\bar{d}$ is a universal property of $\mathcal{M}$.

5. Spectral weights and subclasses of spectral classes

In the singular behaviour of $[C(m^2)]$, inhomogeneities at large scales can also appear in the coefficient of the leading infrared part (3.7). However, following the same steps as the previous section, we will show that once again a splitting of the value of the coefficient corresponds to a macroscopic inhomogeneity of the graph and that a macroscopically homogeneous graph is indeed characterized by universal $\bar{d}$ *and* c. Actually, in this case, the proof is subtler and requires some further mathematical steps.

We first define the spectral *subclasses* of a given spectral class by looking at the spectral weight c_A, proceeding along steps similar to those followed above.

- Let us suppose that, for a given graph $\mathcal{G}$ belonging to the spectral class characterized by $\bar{d}$, there exists at least one non-zero-measure subset $A \subset G$ such that, as $m^2 \to 0$,

$$\mathrm{Sing}[C(m^2)]_A \sim c_A (m^2)^{\bar{d}/2-1} \tag{5.1}$$

  with $c_A \neq c$, with c given as in equation (3.7).
- Then we say that a non-zero-measure subset $M \subset G$, which certainly is maximal with respect to $\bar{d}$, due to its universality, is also maximal with respect to c_A whenever:

  (a) $|M \cap A| > 0$;
  (b) $\mathrm{Sing}[C(m^2)]_M \sim c_M (m^2)^{\bar{d}/2-1}$, with $c_M = c_A$;
  (c) for any non-zero-measure subset $B \subset M$ we have $c_B = c_M$;
  (d) there exists no $B \supset M$ such that $c_B = c_M$ and $|B| > |M|$.

  By this definition it follows that the set of all maximal subsets with respect to c_M coincides with the equivalence class $\{M\}$ and we will call it the *homogeneity subclass* of spectral weight c_M.
- We then prove

  Theorem 2. *The subgraphs $\mathcal{M}$ and its complement $\bar{\mathcal{M}}$ are infinitely separated and define a spectral partition of $\mathcal{G}$.*

- Following the same steps as in the previous section, we then consider a Gaussian model on the graph $\mathcal{M}$ showing that, from the infinite separability of $\mathcal{M}$ and $\bar{\mathcal{M}}$, the coefficient of $\mathrm{Sing}[C(m^2)]_M$ is c_M. Therefore, we can say that c_M is a universal property of the graph $\mathcal{M}$ and defines a *spectral subclass* separated from the rest.

Proof of theorem 2. To prove this theorem we first need the following lemma.

Lemma. *Within a given spectral subclass, for any subset A of the subclass, the asymptotic form of $[C(m^2)]_A$ is invariant under pre-averaging over any normalized point distribution with non-zero-measure support. In other words, if we define*

$$[C(m^2)]_{A,\alpha} = \frac{[\alpha\, C(m^2)]_A}{[\alpha]_A} \tag{5.2}$$

where $\alpha_i > 0$ on a subset of A with non-zero measure, then again

$$\mathrm{Sing}[C(m^2)]_{A,\alpha} \sim c_A (m^2)^{\bar{d}/2-1} \tag{5.3}$$

with no dependence at all for c_A and $\bar{d}$ on the distribution $\alpha = \{\alpha_i;\ i \in A\}$. The proof of this statement is elementary: we define the quantities

$$f_i = (m^2)^{-\bar{d}/2+1} C_{ii}(m^2) - c_A. \tag{5.4}$$

Then, by construction, for any $\epsilon > 0$ there exist a $\delta > 0$ such that we have $|[f]_A| < \epsilon$ as soon as $m^2 < \delta$. Hence we also have

$$\left|[\alpha f]_A\right| < \left(\sup_{i\in A} \alpha_i\right) \left|[f]_A\right| < \left(\sup_{i\in A} \alpha_i\right)\epsilon \tag{5.5}$$

which immediately implies our assertion.

Now we can prove theorem 2.

Let us suppose *ad absurdum* that $\underline{D}(\mathcal{M}, \bar{\mathcal{M}}) = 1$ and therefore that there exists a non-zero-measure subset $\bar{M}' \subset \bar{M}$ such that $\underline{D}(\mathcal{M}, \bar{\mathcal{M}}') = 1$. From the maximality of M it follows that $c_M \neq c_{\bar{M}'}$.

The following proof is given only for $\bar{d} < 4$, owing to brevity and physical requirements. Indeed, a real structure has necessarily a dimension $\bar{d} \leqslant 3$; moreover, from a purely theoretical point of view, the class of models we have in mind, with site variables and link interactions, typically have four as an upper critical dimension for the scaling behaviour.

Let us consider first the case of a spectral class where $[C(m^2)]_G$ diverges when $m^2 \to 0$, that is such that $\bar{d} < 2$. The Schwinger–Dyson equation for $C_{ii}[m^2]$ reads

$$(J_i + m^2\eta_i) C_{ii}(m^2) - \sum_{k\in\mathcal{G}} J_{ik} C_{ki}(m^2) = 1. \tag{5.6}$$

Averaging equation (5.6) over M, we obtain the relation

$$[J\,C]_M + m^2\,[\eta\,C]_M - [J \cdot C]_M = |M| \tag{5.7}$$

where $(J\,C)_i \equiv J_i C_{ii}$, $(\eta\,C)_i \equiv \eta_i C_{ii}$ and $(J \cdot C)_i = \sum_k J_{ik} C_{ki}$. We then divide by $[J\,C]_M$ and let $m^2 \to 0$. Due to the divergence of $[J\,C]_M$ we have that, for any $\epsilon > 0$ there exists a $\delta > 0$ such that, as soon as $m < \delta$,

$$1 - \epsilon \leqslant \frac{[J \cdot C]_M}{[J\,C]_M}. \tag{5.8}$$

Next we set

$$J_{\bar{M}',i} = \sum_{k\in\bar{M}'} J_{ik} \qquad (J \cdot C)_{\bar{M}',i} = \sum_{k\in\bar{M}'} J_{ik} C_{ki} \tag{5.9}$$

and use the positivity of $C_{ii} - C_{ik}$ [2] to push the above inequality to

$$1 - \epsilon \leqslant 1 - \frac{[J_{\bar{M}'}\,C]_M}{[J\,C]_M} + \frac{[(J \cdot C)_{\bar{M}'}]_M}{[J\,C]_M} \tag{5.10}$$

which yields

$$\lim_{m^2\to 0} \frac{[(J \cdot C)_{\bar{M}'}]_M}{[J_{\bar{M}'}\,C]_M} = 1. \tag{5.11}$$

Owing to the symmetry of $\underline{D}(\mathcal{M}, \bar{\mathcal{M}}')$, we may repeat the above steps with M and $\bar{M}'$ interchanged. Since the symmetry of J_{ij} and C_{ij} implies $[(J \cdot C)_{\bar{M}'}]_M = [(J \cdot C)_M]_{\bar{M}'}$, we finally obtain

$$\lim_{m^2\to 0} \frac{[J_{\bar{M}'}\,C]_M}{[J_M\,C]_{\bar{M}'}} = 1. \tag{5.12}$$

At this stage we apply the lemma given above with α identified with $J_{\bar{M}'}$ or J_M, namely

$$[J_{\bar{M}'}\, C]_M \sim c_M\, [J_{\bar{M}'}]_M\, (m^2)^{\bar{d}/2-1} \qquad [J_M\, C]_{\bar{M}'} \sim c_{\bar{M}'}\, [J_M]_{\bar{M}'}\, (m^2)^{\bar{d}/2-1}. \tag{5.13}$$

However, $[J_{\bar{M}'}]_M = [J_M]_{\bar{M}'}$ so that equation (5.12) implies $c_M = c_{\bar{M}'}$, contradicting our initial hypothesis that $\underline{D}(\mathcal{M}, \bar{\mathcal{M}}) = 1$ with M maximal. Hence necessarily $\underline{D}(\mathcal{M}, \bar{\mathcal{M}}) = \infty$, proving our assertion. □

Let us now consider a spectral class where $C(m^2)_G$ does not diverge in the limit $m^2 \to 0$, while its first derivative with respect to m^2, $C'(m^2)_G$, diverges in the same limit. This is the case of a spectral class characterized by a spectral dimension $2 < \bar{d} < 4$, where

$$[C'(m^2)]_{M,\alpha} = \frac{[\alpha\, C'(m^2)]_M}{[\alpha]_M} \sim -(\bar{d}/2-1)\, c_M\, (m^2)^{\bar{d}/2-2} \qquad m^2 \to 0. \tag{5.14}$$

Taking the first derivative with respect to m^2 in the Schwinger–Dyson equation (5.6), we obtain

$$\eta_i C_{ii}(m^2) + m^2 \eta_i C'_{ii}(m^2) = \sum_{k\in\mathcal{G}} J_{ik}[C'_{ki}(m^2) - C'_{ii}(m^2)] \tag{5.15}$$

which can be averaged over M giving

$$[\eta\, C]_M + m^2[\eta\, C']_M = [J \cdot C']_M - [J\, C']_M. \tag{5.16}$$

Together with equation (5.14), this implies

$$\lim_{m^2\to 0} (m^2)^{2-\bar{d}/2}([J \cdot C']_M - [J\, C']_M) = 0^+ \tag{5.17}$$

that is, for any $\epsilon > 0$ there exists a $\delta > 0$ such that, as soon as $m^2 < \delta$

$$0 < \xi([J \cdot C']_M - [J\, C']_M) < \epsilon \tag{5.18}$$

with $\xi = (m^2)^{2-\bar{d}/2}$. This can be rewritten as

$$0 < [(J \cdot C')_M]_M - [J_M\, C']_M + [(J \cdot C')_{\bar{M}}]_M - [J_{\bar{M}}\, C']_M < \xi^{-1}\,\epsilon. \tag{5.19}$$

Now, since $C'_{ij} \equiv -\sum_k \eta_k C_{ik} C_{kj}$ are the elements of a negative semi-definite matrix, one has that $[(J \cdot C')_M]_M - [J_M\, C']_M > 0$. Therefore,

$$0 \leqslant [(J \cdot C')_{\bar{M}}]_M - [J_{\bar{M}'}\, C']_M < \xi^{-1}\,\epsilon. \tag{5.20}$$

Again owing to the symmetry of $\underline{D}(\mathcal{M}, \bar{\mathcal{M}}')$, the previous steps can be repeated with M and $\bar{M}$ interchanged, leading to

$$0 \leqslant [(J \cdot C')_M]_{\bar{M}} - [J_M\, C']_{\bar{M}'} < \xi^{-1}\,\epsilon. \tag{5.21}$$

Since $[(J \cdot C')_{\bar{M}}]_M = [(J \cdot C')_M]_{\bar{M}}$, these two relations imply

$$0 \leqslant |[J_{\bar{M}'}\, C']_M - [J_M\, C']_{\bar{M}'}| < \xi^{-1}\,\epsilon. \tag{5.22}$$

Equation (5.14) entails in the limit $m^2 \to 0$:

$$[J_{\bar{M}'}\, C']_M \sim -(\bar{d}/2-1)\, c_M\, [J_{\bar{M}'}]_M\, \xi^{-1} \qquad [J_M\, C']_{\bar{M}'} \sim -(\bar{d}/2-1)\, c_{\bar{M}'}\, [J_M]_{\bar{M}'}\, \xi^{-1} \tag{5.23}$$

so that, since $[J_{\bar{M}'}]_M = [J_M]_{\bar{M}'}$ from (5.22) one obtains $c_M = c_{\bar{M}'}$, which contradicts our hypothesis $\underline{D}(\mathcal{M}, \bar{\mathcal{M}}') = 1$ and therefore proves our assertion $\underline{D}(\mathcal{M}, \bar{\mathcal{M}}) = \infty$. □

References

[1] Alexander S and Orbach R 1982 *J. Physique Lett.* **43** L625
[2] Hattori K, Hattori T and Watanabe H 1987 *Prog. Theor. Phys. Suppl.* **92** 108
[3] Burioni R and Cassi D 1996 *Phys. Rev. Lett.* **76** 1091
[4] Harary F 1972 *Graph Theory* (Reading, MA: Addison-Wesley)
[5] Woess W 1994 *Bull. London Math. Soc.* **26** 1
Mohar B and Woess W 1989 *Bull. London Math. Soc.* **21** 209
[6] Burioni R, Cassi D and Vezzani A 1999 *Phys. Rev.* E **60** 1500
Burioni R, Cassi D and Vezzani A 1999 *J. Phys. A: Math. Gen.* **32** 5539